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The Uniqueness Regime of Gibbs Fields with Unbounded Disorder[1]

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We consider lattice spin systems with short-range but random and unbounded interactions. We give an elementary proof of uniqueness of Gibbs measures at high temperature or strong magnetic fields, and of the exponential decay of the corresponding quenched correlation functions. The analysis is based on the study of disagreement percolation (as initiated by van den Berg and Maes).

KEY WORDS: Quenched disorder; spin glasses; disagreement percolation; Griffiths' singularities.

1. INTRODUCTION

The subject of this paper is the characterization of the uniqueness regime of Gibbs fields with random potential. We refer to Olivieri *et al*,[(12)] Berretti,[(2)] Fröhlich and Imbrie,[(7)] and Bassalygo and Dobrushin[(1)] for the necessary background. A more recent detailed analysis can be found in Perez,[(13)] Klein,[(11)] and von Dreifus *et al.*[(6)] Here we wish to show how recently developed percolation techniques[(3)] can be applied to give elementary proofs of many results that have appeared in the papers mentioned above.

The extension to interactions with unbounded disorder of general uniqueness criteria such as the Dobrushin[(4)] single-site condition or the Dobrushin–Shlosman[(5)] constructive criteria does not seem straightforward

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at all. The method that we present below and which is based on the uniqueness criterion of van den Berg and Maes[3] provides such a general theory. In another paper[9] we show how the same ideas can be used in the study of dynamics of disordered systems.

2. EQUILIBRIUM STATES WITH RANDOM POTENTIALS

For convenience we consider spin systems defined on the regular d-dimensional lattice $\mathbb{Z}^d$. As will become clear, the arguments that follow are valid on a more general periodic lattice (e.g., the triangular or the FCC lattice). $\mathbb{Z}^d$ comes equipped with the usual structure of nearest neighbor sites x, y connected by bonds $\langle x, y\rangle$. If two sites x and y are nearest neighbors (or adjacent) we will write $x \sim y$.

A configuration σ puts a spin value $\sigma(x) = 1$ or $\sigma(x) = -1$ on every site $x \in \mathbb{Z}^d$. The set $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$ is the set of all configurations. Our results can easily be extended to other finite single-site state spaces.

A probability measure ν on Ω is a Markov field if for every finite $A \subset \mathbb{Z}^d$, $\eta \in \{-1, 1\}^A$,

$$\nu[\sigma = \eta \text{ on } A \mid \sigma(i), i \in A^c] = \nu[\sigma = \eta \text{ on } A \mid \sigma(i), i \in \partial A] \tag{1}$$

where $A^c = \mathbb{Z}^d \backslash A$ and ∂A is the set of all sites in A^c that are adjacent to A. A major problem in statistical mechanics is to determine the Markov fields ν which satisfy for all finite $A \subset \mathbb{Z}^d$, all $\eta \in \{-1, 1\}^A$, $\eta' \in \{-1, 1\}^{\partial A}$

$$\nu[\sigma = \eta \mid \sigma = \eta' \text{ on } \partial A] = Y_A(\eta, \eta') \tag{2}$$

with

$$\{Y_A(\cdot, \eta'), A \subset \mathbb{Z}^d, \text{finite}, \eta' \in \{-1, 1\}^{\partial A}\} \tag{3}$$

a given set of self-consistent conditional probabilities (a specification) possibly parametrized (among other things) by the inverse temperature $\beta \geqslant 0$, external fields, etc.[8] In that case we say that ν is a Gibbs measure with respect to the specification $\{Y_A\}$. We look for conditions on the set $\{Y_A(\cdot, \eta)\}$ such that there exists just one associated Gibbs measure. This is also the context of refs. 3–5.

What is specific to our study here is that the specification is *random*, i.e., the $Y_A = Y_A^\pi$ depend not only on the values of certain fixed parameters, but also on the realization π of the randomness. This is what we call disorder.

One typically considers two types of quenched disorder. One is realized by the nearest neighbor couplings $\{J_{xy}\}_{x\sim y}$ and the other by a set of single-site parameters $\{h_x\}$ (also denoted below by $\{\gamma_x\}$ if not referring to a random magnetic field). We assume that the $\{J_{xy}\}$ are real (possibly infinite)-valued mutually independent, and identically distributed random variables. Similarly for the $\{h_x\}$ ($\{\gamma_x\}$). Examples will follow where these parameters enter explicitly. Sometimes it is, however, more convenient to speak about "realizations" in general without specifying exactly how the disorder is frozen in the interactions or transition rates. Indeed, the relevant objects for our analysis are the (random) specifications and we do not need to refer to specific forms of the interaction. We therefore write π to denote such a general (random) realization (of the disorder).

Π is the set of all these realizations. **Q** is the probability law on the realizations. **E** is the expectation value with respect to the distribution **Q**.

An important example is the following random-field short-range spin glass with formal Hamiltonian:

$$H = -\sum_{\langle x, y\rangle} J_{xy}\sigma(x)\,\sigma(y) - b\sum_x h_x\sigma(x) - h\sum_x \sigma(x) \tag{4}$$

determined by a realization of one- (h_x) and two-point interactions (J_{xy}). The specification $\{Y_A\}$ is obtained by taking the finite-volume Gibbs measures (fixed boundary conditions outside A) with respect to the Hamiltonian H at inverse temperature β. For $A = \{x\}$ we then have (with some abuse of notation)

$$Y_x(\sigma(x), \sigma) = \frac{\exp[\beta\sum_{y\sim x} J_{xy}\sigma(x)\,\sigma(y) + (\beta b h_x + \beta h)\,\sigma(x)]}{Z_x^\beta(\{J_{xy}, \sigma(y)\}_{y\sim x}, h, h_x, b)} \tag{5}$$

Another example is the hard-core lattice gas with random choice of the activities $a_x = \exp(\lambda\gamma_x) - 1$. The $\gamma_x \geqslant 0$ are random and $\lambda \geqslant 0$ is an extra parameter. The single-site conditional distribution is

$$Y_x(1, \eta) = \begin{cases} 1 - e^{-\lambda\gamma_x} & \text{if, for all } y\sim x, \quad \eta(y) = -1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

The construction of a Gibbs measure ν_π with respect to the specification $\{Y_A = Y_A^\pi\}$ will obviously depend on the realization π. The uniqueness of the Gibbs measure should be understood in the sense that with **Q**-probability one there is just one such Gibbs measure. For these equilibrium measures, we define the truncated correlation function of the local functions f and g on Ω as

$$\langle f; g\rangle_\pi = \nu_\pi(fg) - \nu_\pi(f)\,\nu_\pi(g) \tag{7}$$

We also will use the notation

$$\operatorname{dist}(f, g) = \min_{\substack{x \in \operatorname{supp} f \\ x \in \operatorname{supp} g}} |x - y| \tag{8}$$

with $\operatorname{supp} f$ the support of f and $|x - y| = \sum_{\alpha=1}^{d} |x_\alpha - y_\alpha|$. Here $\|f\|$ is the usual supremum norm of f and $\delta_x = \sup_\eta |f(\eta^x) - f(\eta)|$ the oscillation of f at $x \in \mathbb{Z}^d$. The total oscillation is then

$$|||f||| = \sum_{x \in \mathbb{Z}^d} \delta_x f$$

3. UNIQUENESS REGIME

Let

$$q_x = \max_{\eta, \eta'} \operatorname{var}(Y_x(\cdot, \eta), Y_x(\cdot, \eta')) \tag{9}$$

where $\operatorname{var}(\cdot, \cdot)$ ($\in [0, 1]$) is the variational distance.

Everything that follows is expressed in terms of the distribution of the field $\{q_x\}$. Remember that the q_x depend on the realization π and on extra parameters (such as the temperature and external fields) as inherited from the specification. So instead of referring to the high-temperature or strong-external-field regimes separately, the single-phase regime of our disordered system will be obtained if "typically" the q_x are "small" for all x in a sufficiently "big" set. Using definition (9), for every specific model one can get explicit conditions on the realizations and the external parameters. It is important to observe that q_x and q_y may be correlated for $x \neq y$. However, in all relevant examples the randomness in the specification enters locally and has a high degree of independence. While the arguments that follow essentially go through unchanged under the assumption that there is a finite "distance" R for which q_x and q_y are independent whenever the "distance" between x and y exceeds R, for simplicity we require that this already happens for $R = 1$, i.e., q_x and q_y may be correlated for $x \neq y$ only if $x \sim y$, otherwise they are independent; $\{q_x\}$ is a one-dependent random field. This is verified in all examples discussed here.

Another feature present in all our examples of interest is that, with **Q**-probability one, there are finite regions of all sizes on which q_x is large. These regions are responsible for the so-called Griffiths' singularities.[10]

In the spin-glass example (4) with $b=0$, we have

$$q_x = 1/2\left[\tanh\left(\beta \sum_{y\sim x} |J_{xy}| + \beta h\right) + \tanh\left(\beta \sum_{y\sim x} |J_{xy}| - \beta h\right)\right] \tag{10}$$

Note that q_x can be made "small" both by taking $\beta > 0$ small or taking h large. If on the other hand we take in (4) $0 \leqslant J_{xy} = J < \infty$ fixed and $h=0$, we get a random-field Ising model with

$$q_x = \tfrac{1}{2}[\tanh \beta(2dJ + bh_x) + \tanh \beta(2dJ - bh_x)] \tag{11}$$

For the example of the hard-core lattice gas (6), we have $q_x = 1 - \exp(-\lambda y_x)$.

Theorem. If $\{q_x\}_{x\in\mathbb{Z}^d}$, as defined by (9), is a stationary one-dependent random field satisfying.

$$\mathbf{E}(q_x) < \frac{1}{(2d-1)^2} \tag{12}$$

then—with **Q**-probability one—there is a unique Gibbs measure ν_π. Moreover,

$$\mathbf{E}(|\langle f; g\rangle_\pi|) \leqslant C(f,g)\, e^{-m\,\mathrm{dist}(f,g)} \tag{13}$$

for all local functions f and g, with $m>0$ and $C(f,g) = C\,\|f\|\cdot|||g||| < \infty$.

Proof. Absence of independent site percolation with densities $\{q_x\}_{x\in\mathbb{Z}^d}$ implies the uniqueness of the Gibbs state for the specification used in definition (9) for, the $\{q_x\}_{x\in\mathbb{Z}^d}$. This is a consequence of Corollary 2 in van den Berg and Maes.[3] Let ν_π be the unique Gibbs measure. A straightforward application of Corollary 2 in van den Berg and Maes[3] yields that

$$|\langle f; g\rangle_\pi| \leqslant \|f\|\cdot|||g|||\ \max_{x\in \mathrm{supp}\, g}\ \sum_{y\in \mathrm{supp}\, f} G_\pi(x,y) \tag{14}$$

where $G_\pi(x,y)$ is the probability in the independent site percolation process to find an open path from x to y if the realization is π. (Independently a site x is open with probability q_x and is closed with probability $1-q_x$.)

Consider now a self-avoiding path ω, $|\omega|=n$, from x to y. We have

$$G_\pi(x,y) \leqslant \sum_{n\geqslant |x-y|}\ \sum_{|\omega|=n}\ \prod_{i\ \mathrm{even}} q_{\omega_i} \tag{15}$$

where ω_i is the ith site in the path ω. Taking the expectation of both sides of (15) and using the independence of $\{q_x\}_{\{x \text{ even}\}}$ completes the proof

Remarks. 1. From the examples it is clear that it is not always possible to tune an external parameter to make q_x (pointwise) arbitrarily small on any site x. For instance, in the spin-glass example (10) (or in the hard-core lattice gas) always $q_x = 1$ if $J_{xy} = \infty$ ($\gamma_x = \infty$) or in the random-field Ising model (11) $q_x = \tanh 2d\beta J$ when $h_x = 0$, independent of b. When this happens we call a site x "bad." Therefore, a necessary condition for the assumption of the Theorem to be satisfied is that the **Q**-probability of a site to be "bad" is itself small enough. In a way this condition is also sufficient: see, e.g., in the spin-glass example (10), if $\mathbf{Q}\{J_{xy} = \infty\} < 1/(2d-1)^2$, then for $\beta > 0$ sufficiently small condition (12) is satisfied. At the same time, the Theorem does not give the best possible bound on the smallness of $\mathbf{E}(q_x)$. Depending on specific models and using the main underlying idea we can improve substantially on this bound. For example, in the random-field Ising model (11) with $\{h_x\}$ an independent identically distributed field, when $\mathbf{E}(q_x) < p_c(\mathbb{Z}^d)$, the threshold for independent site percolation on $\mathbb{Z}^d$, then the same conclusions as in the Theorem hold.

2. Our main message is that the uniqueness regime of the disordered system will inherit all the nice properties of an associated indepen dent percolation process. We only stated (13) as an important example. Von Dreifus *et al.*[6] show that for the case of the Hamiltonian (4), the exponential decay of the truncated correlation functions [as in (13)] implies the existence of thermodynamic limits and the infinite differentiability of the correlation functions with respect to the external magnetic field h.

3. One can also consider lattices without a bipartite structure, but with a bounded number N of nearest neighbors. Then in every set of M sites we can find at least $M/(N+1)$ nonneighboring sites, where the q_x are independent.

REFERENCES

1. L. A. Bassalygo, and R. L. Dobrushin, Uniqueness of a Gibbs field with random potential—an elementary approach, *Theory Prob. Appl.* **31**:572–589 (1986).
2. A. Beretti, Some properties of random Ising models, *J. Stat. Phys.* **38**:483–496 (1985).
3. J. van den Berg and C. Maes, Disagreement percolation in the study of Markov fields, *Ann. Prob.* **22**:749–763 (1994).
4. R. L. Dobrushin, R L., Description of a random field by means of its conditional probabilities and conditions of its regularity, *Theory Prob. Appl.* **13**:197–224 (1968).

5. R. L. Dobrushin, and S. B. Shlosman, Constructive criterion for the uniqueness of a Gibbs field, in *Statistical Mechanics and Dynamical systems*, J. Fritz, A. Jaffe, and D. Szasz, eds. (Birkhauser, Boston, 1985), pp. 347–370.
6. H. von Dreifus, A. Klein, and J. F. Perez, Taming Griffith's singularities: Infinite differentiability of quenched correlation functions, Preprint (1994).
7. J. Fröhlich, and J. Imbrie, Improved perturbation expansion for disordered systems: beating Griffiths' singularities, *Commun. Math. Phys.* **96**:145–180 (1984).
8. H.-O. Georgii, *Gibbs Measures and Phase Transitions* (De Gruyter, Berlin, 1988).
9. G. Gielis, and J. Maes, Percolation techniques in disordered spin flip dynamics: Relaxation to the unique invariant measure, *Commun. Math. Phys.*, to appear (1995).
10. R. Griffiths, Non-analytic behaviour above the critical point in a random Ising ferromagnet, *Phys. Rev. Lett.* **23**:17–19 (1969).
11. A. Klein, Who is afraid of Griffith's singularities? in *On Three Levels. Micro, Meso and Macroscopic Approaches in Physics*, M. Fannes, C. Maes, and A. Verbeure, eds. (Plenum Press, New York, 1994), pp. 253–258.
12. E. Olivieri, J. F. Perez, and S. G. Rosa, Jr., Some rigorous results on the phase diagram of dilute Ising systems, *Phys. Lett.* **94A**:309 (1983).
13. J. F. Perez, Controlling the effect of Griffith's singularities in random ferromagnets, *Braz. J. Phys.* **23**:356–362 (1993).

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